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Stationary Velocity Distribution in an External Field: A One-Dimensional Model

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The velocity distribution of a charged hard rod coupled to an external field and moving in a neutral equilibrium hard rod gas is studied on the basis of Boltzmann's equation. The exact stationary solution is found. Above a threshold value the field becomes effective in the high-velocity region slowing down the decay of the velocity distribution. The drift velocity and the mean kinetic energy are discussed as functions of the field.

KEY WORDS: Boltzmann equation; external field; threshold value; decay slowing one-dimensional model.

1. INTRODUCTION

Our aim in this note is to determine and analyze the stationary velocity distribution of a charged hard rod coupled to a constant and uniform external field, and moving in a neutral gas composed of mechanically identical rods. The probability density $f(r, v, t)$ for finding the charged test particle at point r with velocity v at time t will be supposed to satisfy the linear Boltzmann equation (see the introduction-discussion in Resibois[(1)]),

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}+a\frac{\partial}{\partial v}\right)f(r, v, t)$$
$$=\rho\int dv'\,|v-v'|\,[f(r, v', t)\,\varphi(v)-f(r, v, t)\,\varphi(v')] \qquad (1.1)$$

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Here a denotes the acceleration resulting from the action of the field. The state of the host neutral fluid is represented by a constant number density ρ and the velocity distribution $\varphi(v)$.

Equation (1.1) describes a one-dimensional Markov process in which the charged particle suffers hard collisions with the frequency proportional to the relative speed $|v-v'|$. The dynamics reduces to instantaneous exchanges of velocities between colliding rods. In the time intervals separating the encounters the charged particle moves with acceleration a.

The complete solution of the initial value problem for equation (1.1) in the absence of the external field was found by Résibois.[1] For $a \neq 0$, two cases have been studied up to now: (i) the zero-temperature limit,[2]

$$\varphi(v) = \delta(v) \tag{1.2}$$

and (ii) the one-speed model,[3]

$$\varphi(v) = \tfrac{1}{2}[\delta(v-u) + \delta(v+u)] \tag{1.3}$$

Our purpose here is to determine the stationary velocity distribution satisfying (1.1) in the physically relevant case where the state of the neutral fluid is described by the Maxwell distribution

$$\varphi(v) = \left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{mv^2}{2k_B T}\right) \tag{1.4}$$

corresponding to some temperature T, k_B is Boltzmann's constant and m the mass of a single rod.

Two characteristic energies appear in this problem:

$ma\rho^{-1}$ = energy absorbed by the charged rod from the field on the mean free path

$k_B T$ = thermal energy of the host fluid.

In order to show explicitly how they enter into the kinetic equation, it is convenient to introduce the dimensionless velocity

$$w = v(k_B T/m)^{-1/2} \tag{1.5}$$

The stationary velocity distribution $F(w)$ satisfies then the equation

$$\varepsilon F'(w) = \Phi(w) \int dw' \, |w-w'| \, F(w') - F(w)\, I(w) \tag{1.6}$$

where F' is the derivative of F,

$$\Phi(w) = \exp\left(-\frac{w^2}{2}\right) \Big/ (2\pi)^{1/2} \tag{1.7}$$

and

$$
\begin{aligned}
I(w) &= \int dw' \, |w - w'| \, \Phi(w') \\
&= |w| + 2\left[\Phi(w) - |w| \int_{|w|}^{\infty} dw' \, \Phi(w') \right]
\end{aligned}
\tag{1.8}
$$

We look for the solution of (1.6) satisfying the conditions

$$
F(w) \geqslant 0, \qquad \int dw \, F(w) = 1 \tag{1.9}
$$

The dimensionless parameter

$$
\varepsilon = \frac{ma\rho^{-1}}{k_B T} \tag{1.10}
$$

measures the ratio of two characteristic energies. One can expect the shape of the stationary distribution to be sensitive to the value of ε. That this is actually the case can be seen by considering equation (1.6) in the asymptotic region $w \to +\infty$. It takes there the simple form

$$
\varepsilon F' = w\Phi - wF \tag{1.11}
$$

so that

$$
F(w) \underset{w \to +\infty}{\simeq} \begin{cases} (1-\varepsilon)^{-1} \Phi(w) + C(\varepsilon) \exp(-w^2/2\varepsilon), & \varepsilon \neq 1 \quad (1.12\text{a}) \\ \dfrac{w^2}{2} \Phi(w), & \varepsilon = 1 \quad (1.12\text{b}) \end{cases}
$$

Hence, when $\varepsilon < 1$, the high-energy asymptotics of F is determined by the thermal bath

$$
F(w) \underset{w \to +\infty}{\simeq} (1-\varepsilon)^{-1} \Phi(w), \qquad \varepsilon < 1 \tag{1.13}
$$

However, when the field is strong enough ($\varepsilon > 1$) the decay of $F(w)$ becomes slower

$$
F(w) \underset{w \to +\infty}{\simeq} C(\varepsilon) \exp(-w^2/2\varepsilon), \qquad \varepsilon > 1 \tag{1.14}
$$

controlled by the field. The existence of the threshold value $\varepsilon = 1$ makes the study of the stationary distribution F quite interesting.

In the next section we solve equation (1.6) and prove the correctness of equations (1.13), (1.14). The stationary drift velocity and the mean kinetic energy are then analyzed as functions of ε. In the last section the discussion of the results is presented.

2. STATIONARY STATE

2.1. General Results

In order to solve equation (1.6) we set

$$F = \Phi G \tag{2.1}$$

where Φ is given by equation (1.7), and rewrite it in the form

$$\varepsilon[G'(w) - wG(w)] = \int dw' \, |w - w'| \, \Phi(w') \, G(w') - G(w) \, I(w) \tag{2.2}$$

Differentiating twice both sides of (2.2) we find

$$\varepsilon[G' - wG]'' = -G''I - 2G'I'$$

which is a second-order differential equation for

$$H = G' \tag{2.3}$$

$$\varepsilon H'' + (I - \varepsilon w) \, H' + 2(I' - \varepsilon) \, H = 0 \tag{2.4}$$

Equation (2.4) can be conveniently written in terms of the function χ related to H by

$$H = \chi \Phi^{-1} \exp\left[-\varepsilon^{-1} \int_0^w dw' \, I(w') \right] \tag{2.5}$$

The substitution (2.5) is suggested by the fact that the function

$$\Phi^{-1} \exp\left[-\varepsilon^{-1} \int_0^w dw' \, I(w') \right]$$

is a solution of the first-order differential equation obtained from (2.2) by neglecting in the right-hand side the gain term. We find

$$\varepsilon(\chi'' + w\chi' - \chi) + (I'\chi - \chi' I) = 0 \tag{2.6}$$

But the definition (1.8) implies the equation

$$I = I'' + wI' \tag{2.7}$$

so that $\chi = I$ is a solution of (2.6), which is quite remarkable. The second linearly independent solution can be constructed by standard methods. In this way, we arrive at the following two solutions of equation (2.6)

$$H_1(w) = I(w)\, \Phi^{-1}(w) \exp\left[-\varepsilon^{-1} \int_0^w dw'\, I(w')\right] \tag{2.8}$$

$$H_2(w) = H_1(w) \int_{-\infty}^{w} dw' \frac{1}{I(w')\, H_1(w')} \tag{2.9}$$

Using equations (1.7), (1.8) we find

$$\int_0^w I(w')\, dw' = (1 + w^2) \int_0^w dw'\, \Phi(w') + w\Phi(w) \tag{2.10}$$

for $w \to -\infty$ we thus get the asymptotic formulas

$$H_1(w) \underset{w \to -\infty}{\simeq} -w \exp[(1 + \varepsilon^{-1})\, w^2/2](2\pi)^{1/2} \tag{2.11}$$

$$H_2(w) \underset{w \to -\infty}{\simeq} \left(\frac{\varepsilon}{1+\varepsilon}\right) w^{-2} \tag{2.12}$$

Equations (2.1), (2.3) show that function H_1, when used to construct the velocity distribution, would lead to an unacceptable behavior $F \sim \exp(w^2/2\varepsilon)$ for $w \to -\infty$. The physically relevant solution of (2.4) is thus H_2. We conclude that the stationary distribution has the form

$$F(w) = \Phi(w)\left[A(\varepsilon) + B(\varepsilon) \int_{-\infty}^{w} dw' H_2(w')\right] \tag{2.13}$$

The constant $A(\varepsilon)$ can be readily calculated from the kinetic equation (1.6) which in the limit $w \to -\infty$ takes the asymptotic form

$$\varepsilon F' = -w\Phi + wF$$

showing that

$$F(w) \underset{w \to -\infty}{\simeq} (1 + \varepsilon)^{-1}\, \Phi$$

Comparison with (2.13) gives

$$A(\varepsilon) = (1 + \varepsilon)^{-1} \tag{2.14}$$

The value of $B(\varepsilon)$ can be found from the normalization condition (1.9)

$$B(\varepsilon) = \left(\frac{\varepsilon}{1+\varepsilon}\right)\left[\int_{-\infty}^{+\infty} dw\, \Phi(w) \int_{-\infty}^{w} dw'\, H_2(w')\right]^{-1} \tag{2.15}$$

The solution (2.13) of equation (1.6) representing the stationary velocity distribution of the charged rod has been thus completely determined.

It turns out that $B(\varepsilon)$ can be given a direct physical interpretation. Indeed, according to equation (2.13) the drift velocity of the charged rod equals

$$\langle w \rangle = \int dw\, wF(w) = B(\varepsilon)\, J(\varepsilon) \tag{2.16}$$

where

$$J(\varepsilon) = \int_{-\infty}^{\infty} dw\, w\Phi(w) \int_{-\infty}^{w} dw' H_2(w') \tag{2.17}$$

As $w\Phi = -\Phi'$, integration by parts yields

$$J(\varepsilon) = \int dw\, \Phi H_2 = \int dw\, I\Gamma \int_{-\infty}^{w} dw' \frac{\Phi}{I^2 \Gamma}$$

where

$$\Gamma(w) = \exp\left[-\varepsilon^{-1} \int_0^w dw'\, I(w')\right] \tag{2.18}$$

But $I\Gamma = -\varepsilon\Gamma'$. Hence, further integration by parts yields

$$J(\varepsilon) = \varepsilon \int_{-\infty}^{\infty} dw \frac{\Phi}{I^2}$$

The above integral can be evaluated with the use of the relation $2\Phi = I'' = I - wI'$ [see equation (2.7)]:

$$\int_{-\infty}^{+\infty} dw \frac{\Phi}{I^2} = \frac{1}{2} \int_{-\infty}^{+\infty} dw \left(\frac{w}{I}\right)' = 1 \tag{2.19}$$

The simple formula then follows

$$\langle w \rangle = \varepsilon B(\varepsilon) \tag{2.20}$$

Up to the factor ε, $B(\varepsilon)$ represents the drift velocity.

2.2. Study of the Three Cases ($\epsilon < 1$, $\epsilon = 1$, $\epsilon > 1$)

2.2.1. When $0 < \varepsilon < 1$, equation (2.15) defining $B(\varepsilon)$ can be simplified. Indeed, equations (1.8) and (2.8)–(2.10) show that in this case

$$H_2(w) \underset{w \to +\infty}{\simeq} \left(\frac{\varepsilon}{1-\varepsilon}\right) w^{-2} \tag{2.21}$$

so that the integral $\int_{-\infty}^{+\infty} dw\, H_2(w)$ is convergent. Using the relation $2\Phi = I''$ we can thus write

$$\frac{2\varepsilon}{1+\varepsilon} B^{-1}(\varepsilon) = \int_{-\infty}^{+\infty} dw\, I'' \int_{-\infty}^{w} dw'\, H_2$$
$$= \int dw\, H_2 - \int_{-\infty}^{+\infty} dw\, I'H_2 \tag{2.22}$$

The differential equation (2.4), satisfied by H_2, integrated over the interval $(-\infty, +\infty)$ shows that

$$\int dw(I - \varepsilon w)\, H_2' + 2 \int dw(I' - \varepsilon)\, H_2 = 0$$

or, after integrating by parts

$$\int dw\, I'H_2 = \varepsilon \int dw\, H_2$$

Inserting this into equation (2.22) we obtain

$$B(\varepsilon) = 2\varepsilon \left[(1 - \varepsilon^2) \int_{-\infty}^{+\infty} dw\, H_2(w) \right]^{-1} \tag{2.23}$$

and for $0 < \varepsilon < 1$ the stationary distribution (2.13) takes the form

$$F(w) = \frac{\Phi(w)}{1+\varepsilon} \left[1 + \frac{2\varepsilon}{1-\varepsilon} \frac{\int_{-\infty}^{w} dw'\, H_2(w')}{\int_{-\infty}^{+\infty} dw'\, H_2(w')} \right] \tag{2.24}$$

When $w \to -\infty$ we recover the asymptotic formula (1.13):

$$F(w) \simeq (1 - \varepsilon)^{-1}\, \Phi(w)$$

In order to evaluate the drift velocity in the weak field limit $\varepsilon \ll 1$, we have to investigate the ε dependence of the integral

$$\int dw\, H_2 = -\varepsilon \int_{-\infty}^{\infty} dw\, \Phi^{-1} \Gamma' \int_{-\infty}^{w} dw'\, \Phi I^{-2} \Gamma^{-1}$$
$$= \varepsilon \int dw \frac{w}{I} H_2 + \varepsilon \int dw\, I^{-2}$$

[function Γ has been defined in equation (2.18)]. As $|w/I| < 1$, it implies the asymptotic formula

$$\int dw\, H_2 \underset{\varepsilon \to 0}{\simeq} \varepsilon \int dw\, I^{-2}$$

Therefore the weak field limit for the drift velocity (2.20) reads

$$\langle w\rangle \underset{\varepsilon \ll 1}{\simeq} 2\varepsilon\left[\int_{-\infty}^{+\infty} dw\, I^{-2}(w)\right]^{-1} \tag{2.25}$$

The stationary current is proportional to the strength of the field. The linear response theory applies in this case.

2.2.2. Let us analyze now the special case $\varepsilon = 1$. As

$$\lim_{w\to+\infty} \Phi^{-1}\Gamma\Big|_{\varepsilon=1} = \left(\frac{2\pi}{e}\right)^{1/2}$$

we get

$$H_2(w) \underset{w\to+\infty}{\simeq} w\left(\frac{2\pi}{e}\right)^{1/2}\int_{-\infty}^{\infty}\frac{dw\,\Phi}{I^2\Gamma} \tag{2.26}$$

In contradistinction to the case $\varepsilon < 1$, function H_2 is not integrable. Putting $\varepsilon = 1$ into the differential equation (2.4) and integrating over the interval $(-\infty, w)$ we find

$$H_2' + (I - w)\,H_2 + \int_{-\infty}^{w} dw'(I' - 1)\,H_2 = 0$$

When $w\to+\infty$, it becomes

$$\lim_{w\to+\infty} H_2'(w) = H_2'(\infty) = \int_{-\infty}^{+\infty} dw'(1 - I')\,H_2$$

$$= \int_{-\infty}^{+\infty} dw\, I'' \int_{-\infty}^{w} dw'\, H_2 \quad .$$

Combining this with equations (2.26), (2.15) leads to the formula

$$B^{-1}(1) = H_2'(\infty) = \left(\frac{2\pi}{e}\right)^{1/2}\int_{-\infty}^{+\infty}\frac{dw\Phi}{I^2\Gamma} \tag{2.27}$$

The stationary distribution for $\varepsilon = 1$ has the form

$$F(w) \underset{\varepsilon=1}{=} \Phi(w)\left[\frac{1}{2} + \int_{-\infty}^{w} dw'\, H_2(w')/H_2'(\infty)\right] \tag{2.28}$$

when $w\to+\infty$, the right-hand side behaves like $w^2\Phi(w)/2$, in accordance with equation (1.12b).

2.2.3. Let us finally study the region $\varepsilon > 1$.

Equations (1.8), (2.8)–(2.10) imply here the asymptotic behavior

$$\int_{-\infty}^{w} dw\, H_2(w') \underset{w \to +\infty}{\simeq} (2\pi)^{1/2} \left(\frac{\varepsilon}{\varepsilon - 1}\right) \exp\left[\frac{(\varepsilon - 1)\, w^2 - 1}{2\varepsilon}\right] \int_{-\infty}^{\infty} dw' \frac{\Phi}{I^2 \Gamma}$$

Therefore the stationary distribution (2.13) decays in the large energy region as

$$\left.\begin{aligned} &F(w) \underset{w \to \infty}{\simeq} C(\varepsilon) \exp(-w^2/2\varepsilon) \\ &\text{with} \quad C(\varepsilon) = B(\varepsilon) \frac{\varepsilon e^{-1/2\varepsilon}}{(\varepsilon - 1)} \int_{-\infty}^{+\infty} dw \frac{\Phi}{I^2 \Gamma} \end{aligned}\right\} \tag{2.29}$$

[compare with equation (1.14)]. The useful representation of $B(\varepsilon)$ is obtained from (2.15) by writing $\Phi = (I' - 1)'/2$, and integrating by parts

$$\left(\frac{\varepsilon}{1 + \varepsilon}\right) B^{-1}(\varepsilon) = \frac{1}{2} \int dw (1 - I')\, H_2 \tag{2.30}$$

We shall now evaluate $B(\varepsilon)$, and thus the drift velocity, in the strong field limit $\varepsilon \to \infty$. To this end, it is convenient to transform the right-hand side of (2.30) by putting in H_2 written as

$$H_2 = (I' H_2 / I)' + \varepsilon^{-1} I' H_2 - I' I^{-2}$$

Integrating by parts the contribution corresponding to the first of the three terms we get

$$\begin{aligned} \left(\frac{2\varepsilon}{1 + \varepsilon}\right) B^{-1}(\varepsilon) = &\int dw \frac{I'' I'}{I} H_2 + \varepsilon^{-1} \int dw (1 - I')\, I' H_2 \\ &+ \int dw \left(\frac{I'}{I}\right)^2 \end{aligned}$$

as $|I'| < 1$, the second term, multiplied by ε^{-1} can be neglected in the $\varepsilon \to \infty$ limit. The last term being ε independent, we focus our attention on the first one which equals

$$\begin{aligned} 2 \int_{-\infty}^{\infty} &dw I'(w) \exp\left[-\int_0^w dw' \frac{I(w')}{\varepsilon}\right] \\ &\times \int_{-\infty}^{w} dw' \frac{\Phi(w')}{I^2(w')} \exp\left[\int_0^{w'} dw'' \frac{I(w'')}{\varepsilon}\right] \end{aligned} \tag{2.31}$$

The contribution to the above expression coming from the integration over the interval $(-\infty, 0)$ satisfies the inequality

$$\left| \int_{-\infty}^{0} dw\, I'\Gamma \int_{-\infty}^{w} dw' \frac{\Phi}{I^2} \Gamma^{-1} \right| < \int_{-\infty}^{0} dw \int_{-\infty}^{w} dw' \frac{\Phi}{I^2}$$

with an ε-independent bound, which is a straightforward consequence of the inequalities

$$|I'| < 1, \qquad \exp\left[-\varepsilon^{-1} \int_{w'}^{w} dw''\, I(w'') \right] < 1 \qquad \text{for} \quad w > w'$$

In order to investigate the integral over the positive semiaxis $w > 0$, we use the fact that the function

$$\Gamma(w) = \exp\left[-\varepsilon^{-1} \int_{0}^{w} dw'\, I(w') \right]$$
$$= \exp\left\{ -\varepsilon^{-1} \left[\frac{1+w^2}{2} - (1+w^2) \int_{w}^{\infty} dw' \Phi(w') + w\Phi(w) \right] \right\}$$

satisfies there the inequalities

$$e^{-\lambda/\varepsilon} \exp(-w^2/2\varepsilon) < \Gamma(w) < \exp(-w^2/2\varepsilon)$$

where

$$\lambda \equiv \sup_{w \geqslant 0} \left[\tfrac{1}{2} - (1+w^2) \int_{w}^{\infty} dw'\, \Phi(w') + w\Phi(w) \right]$$

Clearly in the $\varepsilon \to \infty$ limit, $\Gamma(w)$ for $w > 0$ can be replaced by $\exp(-w^2/2\varepsilon)$.

Taking this into account and putting $w = \sqrt{\varepsilon}\, u$, we rewrite the analyzed contribution to the term (2.31) in the form

$$2\sqrt{\varepsilon} \int_{0}^{\infty} du\, I'(\sqrt{\varepsilon}\, u) \exp\left(-\frac{u^2}{2} \right) \int_{-\infty}^{\sqrt{\varepsilon}\, u} dw' \frac{\Phi}{I^2 \Gamma}$$

When $\varepsilon \to \infty$, we asymptotically obtain

$$2\sqrt{\varepsilon} \int_{0}^{\infty} du\, I'(\infty) \exp\left(-\frac{u^2}{2} \right) \int_{-\infty}^{\infty} dw \frac{\Phi}{I^2} = (2\pi\varepsilon)^{1/2}$$

as $I'(\infty) = 1$, and $\int_{-\infty}^{\infty} dw(\Phi/I^2) = 1$ [see equation (2.19)].

Going back to the drift velocity, we get

$$\langle w \rangle = \varepsilon B(\varepsilon) \underset{\varepsilon \to \infty}{\simeq} (2\varepsilon/\pi)^{1/2} \tag{2.32}$$

The drift velocity in the strong field limit is proportional to the square root of the field.

Before ending this section, let us comment on the ε dependence of the mean kinetic energy of the charged rod. A straightforward calculation yields the formula

$$\begin{aligned}\langle w^2 \rangle &= \int dw\, w^2 F(w) \\ &= 1 + \varepsilon B(\varepsilon) \int_{-\infty}^{+\infty} dw\, \Gamma(w) \int_{-\infty}^{w} dw' \frac{\Phi(w')}{\Gamma^2(w')\, \Gamma(w')}\end{aligned} \tag{2.33}$$

Performing the asymptotic analysis along the same lines as for the drift velocity, one obtains the following results: in the weak-field limit

$$\langle w^2 \rangle \underset{\varepsilon \to 0}{\simeq} 1 + \tfrac{1}{2}\varepsilon^2 \tag{2.34}$$

in the strong-field limit

$$\langle w^2 \rangle \underset{\varepsilon \to \infty}{\simeq} \varepsilon \tag{2.35}$$

3. DISCUSSION

One could expect on physical grounds that the main effect of the coupling between the charged rod and the field should consist in shifting its velocity distribution in the field direction. It turns out that this phenomenon is characterized by a well-defined threshold value of the ratio between two characteristic energies

$$\varepsilon = \frac{ma\rho^{-1}}{k_B T}\bigg|_{\text{threshold}} = 1 \tag{3.1}$$

When $\varepsilon < 1$ (weak field, high-temperature), the large-velocity asymptotics of the stationary distribution $F(w)$ is determined by the heat bath: $F(w) \sim \Phi(w)$. When ε becomes larger than 1 (strong field, low-temperature) the field wins over the thermal bath and slows down the decay of $F(w)$ according to the formula $F(w) \sim \exp(-w^2/2\varepsilon)$, $w \to \infty$.

In this sense the shifting action of the field on the stationary distribution becomes globally effective above the threshold (3.1). This kind of transition does not occur in the two-speed model (1.3) mentioned in the Introduction. Collisions with neutral rods give there the charged particle one of the two velocities u or $-u$, and thus cannot influence the high-energy tail of its velocity distribution.

An interesting problem left open in this paper is that of the way in which the stationary state (2.13) is approached in the course of the evolution described by the Boltzmann equation (1.1). Another question which should be answered is to what extent the properties of the stationary distributions derived here persist in higher dimensions. It seems that the weak- and strong-field dependence of the drift velocity (linear response, Eq. (2.25) and proportionality to the square root of the field equation (2.32)) is not changed when $d > 1$.

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